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Quaternionic metrics and SU(2) Yang–Mills†

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Abstract. A Hermitian, quaternionic metric is constructed in which the real part is the usual metric of a curved space–time and the purely quaternionic part is an (anti) self-dual SU(2) Yang–Mills field. Demanding that a potential exists for the SU(2) field implies that Einstein's field equations with a cosmological constant are satisfied.

1. Introduction

In 1948 Einstein developed a generalised theory of gravitation which combined electromagnetism and gravitation (Einstein 1948). He considered a complex, Hermitian metric whose real part was the usual metric of a curved four-dimensional space–time and whose imaginary part was the field tensor for an electromagnetic field, and consequently showed that, with certain conditions on the Christoffel symbols, Maxwell's equations for the electromagnetic field in the curved space–time were automatically satisfied. It has been suggested by Gürsey (1979) that a possible extension of this method to SU(2) Yang–Mills would involve looking at a quaternionic, Hermitian metric and taking the real part to be the metric of a curved space–time and the purely quaternionic part to be a SU(2) Yang–Mills field tensor.

In this paper, a method of constructing such a metric is given in which $F_{\mu\nu}$ is automatically (anti) self-dual (Dolan 1981) and the existence of a potential for $F_{\mu\nu}$ constrains the Riemann tensor of the metric to be that of an Einstein space (i.e. Einstein's equations with a cosmological constant follow from the existence of a potential). Also, for a self-dual (anti- self-dual) $F_{\mu\nu}$, the Weyl tensor must be anti-self-dual (self-dual).

In § 2, notation and conventions are set up and the properties of quaternions are summarised. In § 3 the conditions on the Riemann and Weyl tensors are derived and in § 4 the results are applied to $\mathbb{C}P^2$ to obtain a self-dual SU(2) field over $\mathbb{C}P^2$ with non-integer topological charge (Atiyah *et al* 1978, Gibbons and Pope 1978). Finally, in § 4 the results are summarised.

2. Notation and conventions

The quaternions or hypercomplex numbers $\mathbb{H}$ have three basis elements, satisfying the algebra

$$e_1^2 = e_2^2 = e_3^2 = e_1 e_2 e_3 = -1. \tag{1}$$

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They can be represented by 2×2 anti-Hermitian matrices $e_a=-\mathrm{i}\sigma_a$, where σ_a are the Pauli matrices. Using these, together with the unit matrix e_0, any element of the quaternions can be written as

$$q=q_ie_i \tag{2}$$

where q_i, $i=0,\ldots,3$ are real numbers. Under multiplication, the quaternions of unit magnitude ($q_iq_i=1$) form the group SU(2). The basis $\{e_i\}$, $i=0,\ldots,3$ satisfies the following identities

$$\begin{aligned} e_ie_j^\dagger-e_je_i^\dagger&=\tfrac{1}{2}\varepsilon_{ijkl}(e_ke_l^\dagger-e_le_k^\dagger)=-2e_a\eta_{ij}^{(+)a}\\ e_i^\dagger e_j-e_j^\dagger e_i&=-\tfrac{1}{2}\varepsilon_{ijkl}(e_k^\dagger e_l-e_l^\dagger e_k)=-2e_a\eta_{ij}^{(-)a} \end{aligned} \tag{3}$$

where

$$\eta_{ij}^{(\pm)a}=\varepsilon_{0aij}\mp\delta_{ia}\delta_{0j}\pm\delta_{ja}\delta_{0i} \tag{4}$$

($a=1,2,3$; $i,j=0,1,\ldots,3$) are the symbols introduced by 't Hooft (1976). The η symbols satisfy the following useful relations

$$\begin{aligned} &\eta_{mn}^{(\pm)a}=\pm\tfrac{1}{2}\varepsilon_{mnkl}\eta_{kl}^{(\pm)a} \qquad \eta_{mn}^{(\pm)a}\eta_{kl}^{(\pm)a}=\delta_{mk}\delta_{nl}-\delta_{ml}\delta_{nk}\pm\varepsilon_{mnkl}\\ &\varepsilon_{abc}\eta_{mn}^{(\pm)b}\eta_{kl}^{(\pm)c}=\eta_{nl}^{(\pm)a}\delta_{mk}+\eta_{mk}^{(\pm)a}\delta_{nl}-\eta_{ml}^{(\pm)a}\delta_{nk}-\eta_{nk}^{(\pm)a}\delta_{ml}. \end{aligned} \tag{5}$$

A point in four-dimensional Euclidean space-time can be represented by a quaternion

$$x=x_ie_i. \tag{6}$$

In what follows, Latin indices will be used to label tensor components in flat-space coordinates, and Greek indices for components in curvilinear coordinates. There is no distinction between covariant and contravariant Latin indices, but Greek indices must be raised and lowered by a metric $g_{\mu\nu}$, with signature (++++).

It is sometimes convenient to split quaternions up into their real and purely quaternion parts

$$\mathrm{Re}\,q=q_0e_0 \qquad \mathrm{Vec}\,q=q_1e_1+q_2e_2+q_3e_3. \tag{7}$$

The action for SU(2) Yang–Mills coupled to gravity is (Charap and Duff 1977)

$$S=\frac{1}{4\kappa}\int_M\sqrt{g}(R-2\Lambda)\mathrm{d}^4x+\text{surface term}-\frac{1}{2e^2}\int_M\sqrt{g}\,\mathrm{Tr}\{F^{\mu\nu}F_{\mu\nu}\}\,\mathrm{d}^4x \tag{8}$$

where $F_{\mu\nu}$ is purely quaternionic. Here Λ is a cosmological constant, $\kappa=4\pi G$, R is the curvature scalar for the manifold M and e is the Yang–Mills coupling constant. The surface term is only present for manifolds which are non-compact, or have a boundary (Gibbons and Hawking 1977). In what follows only compact manifolds without a boundary will be considered. $F_{\mu\nu}$ is derivable from a purely quaternionic potential $A_\mu=-\mathrm{i}\sigma_aA_\mu{}^a$ via

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu]. \tag{9}$$

In the curved space–time described by $g_{\mu\nu}$, the (anti) self-duality equations for $F_{\mu\nu}$ take the form

$$F^{\mu\nu}=\pm\frac{1}{2}\frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{g}}F_{\rho\sigma} \tag{10}$$

where $\varepsilon^{0123}=+1$ in any coordinate system. If (10) is satisfied then the Yang–Mills equations obtained from (8) by varying A_μ are automatically satisfied via Bianchi's identity for $F_{\mu\nu}$. Also, if (10) holds, the energy–momentum tensor for $F_{\mu\nu}$ vanishes, thus, for (anti) self-dual $F_{\mu\nu}$, Einstein's field equations become the vacuum equations

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}(R-2\Lambda)=0 \tag{11}$$

and, for non-zero R, a cosmological constant is necessary.

The topological charge for $F_{\mu\nu}$ is

$$Q=-\frac{1}{16\pi^2}\int_M \frac{\varepsilon^{\mu\nu\rho\sigma}}{2\sqrt{g}}\operatorname{Tr}(F_{\mu\nu}F_{\rho\sigma})\sqrt{g}\,d^4x. \tag{12}$$

3. Quaternionic Vierbeins

Let us construct quaternionic Vierbeins

$$h_\mu=h_{i\mu}e_i \qquad \det h_{i\mu}>0 \tag{13}$$

such that the resulting quaternionic metric

$$H_{\mu\nu}=h_\mu h_\nu^\dagger \tag{14}$$

has $g_{\mu\nu}$ for its real part and $F_{\mu\nu}$ for its purely quaternionic part

$$g_{\mu\nu}=\tfrac{1}{2}\operatorname{Tr}H_{\mu\nu} \qquad F_{\mu\nu}=\lambda \operatorname{Vec} h_\mu h_\nu^\dagger. \tag{15}$$

Here λ is a constant which must be included on dimensional grounds. It has dimensions of (length)$^{-2}$. The above construction was first considered by Gürsey *et al* (1979). It is elegant because $F_{\mu\nu}$ is automatically self-dual (Dolan 1981).

$$\begin{aligned} {}^*F^{\mu\nu} &= \frac{1}{2}\frac{\varepsilon^{\mu\nu\alpha\beta}}{\sqrt{g}}F_{\alpha\beta} \\ &= \tfrac{1}{2}\lambda h_i{}^\mu h_j{}^\nu h_k{}^\alpha h_l{}^\beta \varepsilon^{ijkl}\eta_{mn}{}^{(+)a}h_{m\alpha}h_{n\beta}(\mathrm{i}\sigma^a) \\ &= \lambda\,\mathrm{i}\sigma^a\eta_{ij}{}^{(\pm)a}h_i{}^\mu h_j{}^\nu = F^{\mu\nu} \end{aligned} \tag{16}$$

where we have used

$$\frac{\varepsilon^{\mu\nu\alpha\beta}}{\sqrt{g}}=(\det h_{i\gamma})^{-1}\varepsilon^{\mu\nu\alpha\beta}=h_i{}^\mu h_j{}^\nu h_k{}^\alpha h_l{}^\beta\varepsilon_{ijkl} \tag{17}$$

and

$$h_{i\mu}h_j{}^\mu=\delta_{ij}. \tag{18}$$

An anti-self-dual field can be obtained either by using $H_{\mu\nu}=h_\mu{}^\dagger h_\nu$ or by choosing the opposite orientation for the Vierbeins ($\det h_{i\mu}<0$). The choice of Vierbeins is, of course, not unique and is only defined up to a local O(4) rotation on the Latin indices.

The topological charge for the configuration is found from (12) and (5)

$$Q=\pm\frac{3\lambda^2}{2\pi^2}\int_M\sqrt{g}\,d^4x \tag{19}$$

where the upper (lower) sign is for self-dual (anti-self-dual) fields.

The above construction of a self-dual field is, however, of no use unless one can find a potential for $F_{\mu\nu}$, (9). To this end, let us assume that a potential does exist, and invert the equations of motion for $F_{\mu\nu}$ to try to find an A_μ. Varying A_μ in (8) gives

$$\partial_\mu\{\sqrt{g}F^{\mu\nu}\} = \sqrt{g}[F^{\mu\nu}, A_\mu]. \tag{20}$$

Using (15) for $F^{\mu\nu}$, this is

$$\mathrm{i}\,\eta_{ij}{}^{(+)c}\partial_\mu\{hh_i{}^\mu h_j{}^\nu\} = 2\,\mathrm{i}h\varepsilon^{abc}\eta_{ij}{}^{(+)b}A_\mu{}^a h_i{}^\mu h_j{}^\nu \tag{21}$$

where $h = \det h_{i\mu} > 0$. Then using (18) and the properties of the η symbols (5), this can be inverted to find $A_\mu{}^a$ in terms of $h_{i\mu}$

$$\begin{aligned}A_\mu{}^a = (&\eta_{kj}{}^{(+)a}\delta_{li} + \eta_{li}{}^{(+)a}\delta_{kj} + \eta_{ij}{}^{(+)a}\delta_{kl}\\ &- \eta_{ki}{}^{(+)a}\delta_{lj} - \eta_{lj}{}^{(+)a}\delta_{ki})h_{l\mu}h_{k\alpha}\partial_\beta(hh_i{}^\beta h_j{}^\alpha)/4h.\end{aligned} \tag{22}$$

In terms of quaternions, (22) is

$$A_\mu = \tfrac{1}{4}\{\mathrm{Vec}[h_\alpha(\partial_\mu h^\alpha)^\dagger] + g_{\mu\beta}\,\mathrm{Vec}[h^\alpha(\partial_\alpha h^\beta)^\dagger] + \mathrm{Vec}(h^\alpha h_\beta{}^\dagger)\,\mathrm{Re}[(\partial_\mu h_\alpha)h^{\beta\dagger}]\} \tag{23}$$

where a term proportional to $(\partial_\mu \ln h + h_{j\nu}\partial_\mu h_j{}^\nu)$ has been dropped from (22) by use of the identity $\ln(\det h_{i\mu}) = \mathrm{Tr}\ln(h_{i\mu})$.

The simple case of S^4 wrapped round itself k times, for which

$$h_\mu = \frac{\partial_\mu(x^k)^\dagger}{(1+|x|^2)} \tag{24}$$

and

$$A_\mu = \frac{1}{2}\,\frac{(x^k)^\dagger\partial_\mu x^k - (\partial_\mu x^{k\dagger})x^k}{(1+|x|^2)}, \tag{25}$$

has been considered by Gürsey *et al* (1979).

In terms of the Christoffel symbols $\Gamma_{\mu\nu}{}^\alpha$ derived from the metric in the usual way, equation (23) reduces to

$$\begin{aligned}A_\mu &= \tfrac{1}{4}\{\mathrm{Vec}[h_\alpha(\partial_\mu h^\alpha)^\dagger] + \Gamma_{\beta\mu}{}^\alpha\,\mathrm{Vec}(h_\alpha h^{\beta\dagger})\}\\ &= \tfrac{1}{4}\,\mathrm{Vec}[h_\alpha(h^\alpha{}_{;\mu})^\dagger]\end{aligned} \tag{26}$$

where the semi-colon denotes covariant differentiation. Similarly, a potential for an anti-self-dual field is given by

$$A_\mu = \tfrac{1}{4}\,\mathrm{Vec}(h_\alpha{}^\dagger h^\alpha{}_{;\mu}). \tag{27}$$

Each of these lies in the algebra of SU(2), and so their direct sum lies in a complex representation of the algebra of O(4). Denoting the self-dual (anti-self-dual) SU(2) potentials by the superscript (±), an O(4) potential is

$$\Gamma_\mu = \begin{pmatrix} A_\mu{}^{(+)} & 0 \\ 0 & A_\mu{}^{(-)} \end{pmatrix} \tag{28}$$

where each entry in (28) is a 2×2 matrix. If we denote the generators of the complex representation of O(4) by $\sigma_{ij} = -\sigma_{ji}$, then

$$\Gamma_\mu = \tfrac{1}{2}\sigma_{ij}\Gamma_\mu{}^{ij} = \tfrac{1}{2}\sigma_{ij}h_{i\alpha}h_j{}^\alpha{}_{;\mu} \tag{29}$$

and Γ_μ is seen to be the spin connection for the manifold. Thus the self-dual SU(2) field constructed in (15) is that obtained by projecting the O(4) spin connection onto a SU(2) subgroup. This procedure has been used by Charap and Duff (1977) to construct self-dual, SU(2), Yang–Mills fields in spaces for which $R_{\mu\nu} = 0$. As noted earlier, a different choice of Vierbeins, $h_{i\mu}$, will lead to the same metric if they are related by an O(4) gauge transformation on the i index. We see from equation (28) that this will induce an SU(2) gauge transformation on each potential $A_\mu^{(\pm)}$.

The SU(2) field tensor derived from (27) via (9) is found to be

$$F_{\mu\nu} = \tfrac{1}{4}\,\mathrm{Vec}(h^\sigma h^{\rho\dagger})R_{\sigma\rho\mu\nu} \tag{30}$$

and similarly an anti-self-dual field is given by changing the Hermitian conjugate from the second to the first Vierbien on the right-hand side. Again the self-dual and anti-self-dual SU(2) field tensors can be combined in a direct sum to yield a complex representation of an O(4) field tensor, which is exactly that of Utiyama (1956) in his O(4) gauge theory of gravity. The approach of Charap and Duff (1977) was to use the fact that, if $R_{\mu\nu} = 0$, then the Riemann tensor is double self-dual and the SU(2) field (3) is automatically self-dual. Equation (15) was not assumed. Our approach is to assume equation (15) and then derive conditions that the Riemann tensor must satisfy in order that a potential exists. From equations (15) and (30)

$$\lambda\,\mathrm{Vec}(h_\mu h_\nu^\dagger) = \tfrac{1}{4}\,\mathrm{Vec}(h^\sigma h^{\rho\dagger})R_{\sigma\rho\mu\nu}. \tag{31}$$

Contracting $(\mathrm{Vec}\,H_{\mu\nu})$ with both sides of (31) gives

$$\begin{aligned}
&\lambda \eta_{ij}^{(+)a}\eta_{kl}^{(+)a} h_{i\mu}h_{j\nu}h_k{}^\mu h_l{}^\nu \\
&\qquad = \tfrac{1}{4}\eta_{ij}^{(+)a}\eta_{kl}^{(+)a} h_i{}^\sigma h_j{}^\rho h_k{}^\mu h_l{}^\nu R_{\sigma\rho\mu\nu} \\
&\qquad \Leftrightarrow \lambda\,(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + \varepsilon_{ijkl})\delta_{ik}\delta_{jl} \\
&\qquad = \tfrac{1}{4}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + \varepsilon_{ijkl}) h_i{}^\sigma h_j{}^\rho h_k{}^\mu h_l{}^\nu R_{\sigma\rho\mu\nu} \\
&\qquad \Leftrightarrow \lambda = R/4!
\end{aligned} \tag{32}$$

since $\varepsilon^{\sigma\rho\mu\nu}R_{\sigma\rho\mu\nu} = 0$. Thus *the curvature scalar must be a non-zero constant.* This result has been deduced directly from (15) and the assumption that a potential exists. Exactly the same result is obtained by using an anti-self-dual field.

Furthermore, if we square both sides of (31), we find (using (32))

$$\begin{aligned}
&\frac{R^2}{36}\eta_{ij}^{(+)a}\eta_{kl}^{(+)a} h_{i\mu}h_{j\nu}h_k{}^\mu h_l{}^\nu \\
&\qquad = \eta_{ij}^{(+)a}\eta_{kl}^{(+)a} h_i{}^\sigma h_j{}^\rho h_k{}^\alpha h_l{}^\beta R_{\sigma\rho\mu\nu}R_{\alpha\beta}{}^{\mu\nu} \\
&\qquad \Leftrightarrow \frac{R^2}{6} = \frac{1}{2}\left(R_{\sigma\rho\mu\nu} + \frac{\varepsilon_{\sigma\rho\alpha\beta}}{2\sqrt{g}}R^{\alpha\beta}{}_{\mu\nu}\right)\left(R^{\sigma\rho\mu\nu} + \frac{\varepsilon^{\sigma\rho\lambda\tau}}{2\sqrt{g}}R_{\lambda\tau}{}^{\mu\nu}\right).
\end{aligned} \tag{33}$$

For an anti-self-dual field the ε tensors change sign. Decomposing the Riemann tensor into R, $R_{\mu\nu}$ and the Weyl tensor in the usual way

$$R_{\rho\sigma\mu\nu} = C_{\rho\sigma\mu\nu} + \tfrac{1}{2}(g_{\rho\mu}R_{\sigma\nu} + g_{\sigma\nu}R_{\rho\mu} - g_{\rho\nu}R_{\sigma\mu} - g_{\sigma\mu}R_{\rho\nu}) - \tfrac{1}{6}R\,(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}) \tag{34}$$

equation (33) becomes

$$R^2 - 4R_{\mu\nu}R^{\mu\nu} = \left(C_{\mu\nu\lambda\rho} \pm \frac{1}{2}\,\frac{\varepsilon_{\lambda\rho\sigma\gamma}}{\sqrt{g}}C_{\mu\nu}{}^{\sigma\gamma}\right)\left(C^{\mu\nu\lambda\rho} \pm \frac{1}{2}\,\frac{\varepsilon^{\lambda\rho\alpha\beta}}{\sqrt{g}}C^{\mu\nu}{}_{\alpha\beta}\right) \tag{35}$$

where the upper sign holds for self-dual $F_{\mu\nu}$ and the lower sign for anti-self-dual $F_{\mu\nu}$.

We now have two conditions on the Riemann tensor (32) and (35), and one has been used to eliminate λ. A third condition can be obtained by iterating equation

(30) twice and using the duality properties of $F_{\mu\nu}$.

$$F_{\mu\nu}=\frac{6}{R}F^{\rho\sigma}R_{\rho\sigma\mu\nu}=\pm\frac{3}{R}\frac{\varepsilon^{\rho\sigma\alpha\beta}}{\sqrt{g}}F_{\alpha\beta}R_{\rho\sigma\mu\nu}$$
$$=\pm\frac{18}{R^2}\frac{\varepsilon^{\rho\sigma\alpha\beta}}{\sqrt{g}}F^{\lambda\tau}R_{\lambda\tau\alpha\beta}R_{\rho\sigma\mu\nu}$$
$$=\frac{9}{R^2}\frac{\varepsilon^{\rho\sigma\alpha\beta}}{\sqrt{g}}\frac{\varepsilon^{\lambda\tau\gamma\delta}}{\sqrt{g}}F_{\gamma\delta}R_{\lambda\tau\alpha\beta}R_{\rho\sigma\mu\nu}. \tag{36}$$

Now contracting both sides of (36) with $F_{\mu\nu}$ yields

$$\frac{R^2}{6}=\frac{1}{4}\frac{\varepsilon^{\rho\sigma\alpha\beta}}{\sqrt{g}}\frac{\varepsilon^{\lambda\tau\mu\nu}}{\sqrt{g}}R_{\lambda\tau\alpha\beta}R_{\rho\sigma\mu\nu}\pm\frac{1}{2}\frac{\varepsilon^{\rho\sigma\alpha\beta}}{\sqrt{g}}R_{\rho\sigma\mu\nu}R_{\alpha\beta}{}^{\mu\nu}. \tag{37}$$

There is a relation between the quadratic invariants of the Riemann tensor (Lanczos 1938). They are not all independent, but must satisfy

$$\frac{1}{4}\frac{\varepsilon^{\rho\sigma\alpha\beta}}{\sqrt{g}}\frac{\varepsilon^{\lambda\tau\mu\nu}}{\sqrt{g}}R_{\rho\sigma\lambda\tau}R_{\alpha\beta\mu\nu}=R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}-4R_{\alpha\beta}R^{\alpha\beta}+R^2. \tag{38}$$

Using (38) in (33) and (37) yields

$$R^2-4R_{\alpha\beta}R^{\alpha\beta}=0. \tag{39}$$

Equation (39) for a positive definite metric is true *if and only if* the metric is Einstein (Petrov 1969). This can be most easily seen by writing (39) as

$$(R_\mu{}^\nu-\tfrac{1}{4}R\delta_\mu{}^\nu)(R_\nu{}^\mu-\tfrac{1}{4}R\delta_\nu{}^\mu)=0. \tag{40}$$

If we choose orthonormal coordinates at any given point, each term on the left-hand side is a perfect square, and they all appear with a positive sign, provided the metric is positive definite, and so each term must be identically zero. Therefore

$$R_{\mu\nu}=\tfrac{1}{4}Rg_{\mu\nu} \tag{41}$$

i.e. the space must be Einstein.

Thus, given the construction (15) of an (anti) self-dual $F_{\mu\nu}$, we have proved that a condition for a potential to exist is that the space must be Einstein, i.e. that Einstein's equations with a cosmological constant *must* be satisfied. Note that this is only true for $R\neq 0$. If $R=0$, then $\lambda=0$ and the construction (15) will not work. However, the identification of a SU(2) potential by projecting out a SU(2) factor of the O(4) spin connection is perfectly valid, even for $R=0$ (Charap and Duff 1977).

Given (41), equation (35) tells us that

$$(C_{\mu\nu\lambda\rho}\pm\tfrac{1}{2}\varepsilon_{\lambda\rho\sigma\gamma}C_{\mu\nu}{}^{\sigma\gamma})\left(C^{\mu\nu\lambda\rho}\pm\frac{1}{2}\frac{\varepsilon^{\lambda\rho\alpha\beta}}{\sqrt{g}}C^{\mu\nu}{}_{\alpha\beta}\right)=0 \tag{42}$$

and the same reasoning as that following equation (40) tells us that, for a self-dual field to exist, the Weyl tensor must be anti-self-dual, and for an anti-self-dual field to exist, the Weyl tensor must be self-dual. Thus the only manifolds, with $R\neq 0$, which admit *both* instantons and anti-instantons are conformally flat, e.g. S^4 and T^4, the four-dimensional torus.

The topological charge of the field configuration (15) can be expressed in terms of the topological invariants of the manifold. These are the Euler characteristic χ

and the Hirzebruch signature τ, which for compact manifolds without boundary are given by

$$\chi = \frac{1}{128\pi^2}\int_M \frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{g}}\frac{\varepsilon^{\alpha\beta\gamma\delta}}{\sqrt{g}} R_{\mu\nu\alpha\beta}R_{\rho\sigma\gamma\delta}\sqrt{g}\,\mathrm{d}^4x$$

$$= \frac{1}{32\pi^2}\int_M (C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \tfrac{1}{6}R^2)\sqrt{g}\,\mathrm{d}^4x \tag{43}$$

$$\tau = \frac{1}{96\pi^2}\int_M \frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{g}} R_{\mu\nu\alpha\beta}R_{\rho\sigma}{}^{\alpha\beta}\sqrt{g}\,\mathrm{d}^4x$$

$$= \frac{1}{96\pi^2}\int_M \frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{g}} C_{\mu\nu\alpha\beta}C_{\rho\sigma}{}^{\alpha\beta}\sqrt{g}\,\mathrm{d}^4x. \tag{44}$$

Since the Weyl tensor must be (anti) self-dual for a potential for $F_{\mu\nu}$ to exist, (43) and (44) give

$$2\chi \pm 3\tau = \frac{1}{96\pi^2}\int_M R^2\sqrt{g}\,\mathrm{d}^4x. \tag{45}$$

Using (19) and (32) this yields

$$Q = \pm\tfrac{1}{2}\chi + \tfrac{3}{4}\tau. \tag{46}$$

For example S^4 has $\chi = 2$ and $\tau = 0$, giving $Q = \pm 1$, and these solutions are the original instantons of Belavin *et al* (1975).

4. $\mathbb{C}P^2$

The compact, four-dimensional space $\mathbb{C}P^2$ has been proposed as a gravitational instanton (Gibbons and Pope 1978, Eguchi and Freund 1976). It has non-zero, constant scalar curvature, admits an Einstein metric, and its Weyl tensor is either self-dual or anti-self-dual, depending on the orientation chosen. SU(2) Yang–Mills over $\mathbb{C}P^2$ has been considered in Atiyah *et al* (1978) and Gibbons and Pope (1978). A metric for $\mathbb{C}P^2$ is given, in complex coordinates y and z, by

$$\mathrm{d}s^2 = \frac{2}{(1+y\bar{y}+z\bar{z})^2}[(1+z\bar{z})\,\mathrm{d}y\,\mathrm{d}\bar{y} + (1+y\bar{y})\,\mathrm{d}z\,\mathrm{d}\bar{z} - y\bar{z}\,\mathrm{d}\bar{y}\,\mathrm{d}z - z\bar{y}\,\mathrm{d}y\,\mathrm{d}\bar{z}] \tag{47}$$

(the dimensions are implicitly fixed by using unity in the denominator). Writing this as a Hermitian (complex) metric (Yano 1965)

$$\mathrm{d}s^2 = 2g_{\alpha\bar{\beta}}\,\mathrm{d}u^\alpha\,\mathrm{d}\bar{u}^\beta \tag{48}$$

where $u^1 = y$, $u^2 = z$, $g_{\alpha\bar{\beta}}$ is given by complex Vierbeins ($\alpha, \beta = 1, 2$)

$$g_{\alpha\bar{\beta}} = (hh^{\dagger})_{\alpha\bar{\beta}} \tag{49}$$

where h has the form ($|x|^2 = y\bar{y} + z\bar{z}$)

$$h_{\beta\bar{\delta}} = \begin{pmatrix} (1-\mathrm{i}z\bar{z})/|x| & \mathrm{i}\bar{y}z/|x| \\ \mathrm{i}y\bar{z}/|x| & (1-\mathrm{i}y\bar{y})/|x| \end{pmatrix}(1+|x|^2)^{-1}. \tag{50}$$

Here, b labels locally flat, complex coordinates ($b = 1, 2$). With

$$u^1 = y = x^0 - \mathrm{i}x^3 \qquad \varepsilon^{0123} = +1 \tag{51}$$
$$u^2 = z = x^2 - \mathrm{i}x^1$$

the metric (47) is found to give rise to an anti-self-dual Weyl tensor, and a constant scalar curvature $R = 12$.

Now apply equation (15) in its complex form ($\lambda = R/4! = \frac{1}{2}$)

$$F_{yz} = \tfrac{1}{4}\{h_y(h_{\bar z})^\dagger - h_z(h_{\bar y})^\dagger\} \tag{52}$$

etc, with

$$h_y = \frac{1}{\sqrt{2}}[(e_0 - \mathrm{i}e_3)h_{y\bar 1} + (e_2 - \mathrm{i}e_1)h_{y\bar 2}]$$

$$h_{\bar y} = \frac{1}{\sqrt{2}}[(e_0 + \mathrm{i}e_3)(\overline{h_{y\bar 1}}) + (e_2 + \mathrm{i}e_1)(\overline{h_{y\bar 2}})] \text{ etc.}$$

The $\sqrt{2}$ is present because of the factor of two in (48). This gives

$$\begin{aligned}
F_{y\bar z} &= -(F_{\bar y z})^\dagger \\
&= \frac{1}{2(1+|x|^2)^2|x|^2}\{\mathrm{i}e_3 z\bar y(z\bar z - y\bar y + 2\mathrm{i}|x|) \\
&\quad - e_2[z\bar z y\bar y + |x|^2 + \mathrm{i}|x|(y\bar y - z\bar z) - z^2\bar y^2] \\
&\quad - \mathrm{i}e_1[z\bar z y\bar y + |x|^2 + \mathrm{i}|x|(y\bar y - z\bar z) + z^2\bar y^2]\}
\end{aligned}$$

$$\begin{aligned}
F_{y\bar y} &= \frac{1}{2(1+|x|^2)^2|x|^2}\{\mathrm{i}e_3[z\bar z(y\bar y - z\bar z) - |x|^2] + e_2[y\bar z(z\bar z + \mathrm{i}|x|) - z\bar y(z\bar z - \mathrm{i}|x|)] \\
&\quad + \mathrm{i}e_1[\bar z y(z\bar z + \mathrm{i}|x|) + z\bar y(z\bar z - \mathrm{i}|x|)]\} \\
F_{z\bar z} &= \frac{1}{2(1+|x|^2)^2|x|^2}\{\mathrm{i}e_3[y\bar y(y\bar y - z\bar z) + |x|^2] + e_2[\bar z y(y\bar y - \mathrm{i}|x|) - z\bar y(y\bar y + \mathrm{i}|x|)] \\
&\quad + \mathrm{i}e_1[\bar z y(y\bar y - \mathrm{i}|x|) + z\bar y(y\bar y + \mathrm{i}|x|)]\} \\
F_{yz} &= F_{\bar y\bar z} = 0.
\end{aligned} \tag{53}$$

It is straightforward to verify that (53) is self-dual, where, for complex coordinates, the self-duality equations are, using (51),

$$F^{z\bar y} = -\frac{F_{y\bar z}}{(\det g_{\alpha\bar\beta})} \qquad F^{y\bar z} = -\frac{F_{z\bar y}}{(\det g_{\alpha\bar\beta})}$$
$$F^{y\bar y} = \frac{F_{z\bar z}}{(\det g_{\alpha\bar\beta})} \qquad F^{yz} = -\frac{F_{\bar y\bar z}}{(\det g_{\alpha\bar\beta})} \qquad F^{\bar y\bar z} = -\frac{F_{yz}}{(\det g_{\alpha\bar\beta})}. \tag{54}$$

(Note that $\sqrt{g} = 4(\det g_{\alpha\bar\beta})$.)

A potential for (53) is given by (27) and found to be

$$\begin{aligned}
A_y = -(A_{\bar y})^\dagger &= \frac{1}{2}\frac{(1+\mathrm{i}/|x|)}{(1+|x|^2)}\{\mathrm{i}e_3\bar y(\tfrac{1}{2} + z\bar z/|x|^2) + \tfrac{1}{2}e_2[\bar z(1 + z\bar z/|x|^2) + z\bar y^2/|x|^2] \\
&\quad + \tfrac{1}{2}\mathrm{i}e_1[\bar z(1 + z\bar z/|x|^2) - z\bar y^2/|x|^2]\}
\end{aligned}$$

$$A_z = -(A_{\bar z})^\dagger = -\frac{1}{2}\frac{(1+\mathrm{i}/|x|)}{(1+|x|^2)}\{\mathrm{i}e_3\bar z(\tfrac{1}{2}+y\bar y/|x|^2)+\tfrac{1}{2}e_2[\bar y(1+y\bar y/|x|^2)+y\bar z^2/|x|^2]$$
$$\tfrac{1}{2}\mathrm{i}e_1[-\bar y(1+y\bar y/|x|^2)+y\bar z^2/|x|^2]\} \tag{55}$$

and (53) is obtained from (55) via (9).

$\mathbb{C}P^2$ has $\chi = 3$ and $\tau = -1$, so (46) yields

$$Q = \tfrac{3}{4} \tag{56}$$

which can be verified directly from (19), since $R = 12$. Thus the field has non-integral topological charge showing it is not globally well defined (Gibbons and Pope 1978).

The anti-self-dual field, obtained from (15) by interchanging h_μ and $h_\nu{}^\dagger$, is

$$\begin{aligned}
F_{y\bar y} &= \mathrm{i}e_3\frac{\lambda(1+z\bar z)}{(1+|x|^2)^2} & F_{z\bar z} &= \mathrm{i}e_3\frac{\lambda(1+y\bar y)}{(1+|x|^2)^2}\\
F_{y\bar z} &= -\mathrm{i}e_3\frac{\lambda\bar y z}{(1+|x|^2)^2} & F_{\bar y z} &= \mathrm{i}e_3\frac{\lambda\bar z y}{(1+|x|^2)^2}\\
F_{yz} &= \frac{\lambda(1-\mathrm{i}|x|)}{(1+|x|^2)^2}(e_2-\mathrm{i}e_1) & F_{\bar y\bar z} &= \frac{\lambda(1+\mathrm{i}|x|)}{(1+|x|^2)^2}(e_2+\mathrm{i}e_1).
\end{aligned} \tag{57}$$

Since the Weyl tensor is not self-dual, we know that no SU(2) potential exists for this configuration (hence λ has not been constrained). If, however, we set $F_{yz} = F_{\bar y\bar z} = 0$, the remaining field components in (57) give an anti-self-dual *abelian* field, which, by inspection, has a potential

$$\begin{aligned}
A_y &= -\frac{\lambda\bar y\mathrm{i}e_3}{2(1+|x|^2)} & A_{\bar y} &= \frac{\lambda y\mathrm{i}e_3}{2(1+|x|^2)}\\
A_z &= -\frac{\lambda\bar z\mathrm{i}e_3}{2(1+|x|^2)} & A_{\bar z} &= \frac{\lambda z\mathrm{i}e_3}{2(1+|x|^2)}.
\end{aligned} \tag{58}$$

In fact, the 2-form obtained from (57) by setting $F_{yz} = F_{\bar y\bar z} = 0$ is just the Kahler 2-form for $\mathbb{C}P^2$ (Gibbons and Pope 1978, Yano 1965). The topological charge of this configuration is

$$Q = -\tfrac{1}{2}\lambda^2 \tag{59}$$

(where e_3 is normalised so that $e_3^2 = -1$, not a matrix) and it has been discussed (Hawking and Pope 1978) in connection with the problem of putting spinor fields on $\mathbb{C}P^2$.

5. Conclusion

It has been shown how to construct a quaternionic metric for which the real part is the usual metric of a curved space and the purely quaternionic part is (anti) self-dual

$$H_{\mu\nu} = h_\mu h_\nu{}^\dagger. \tag{60}$$

Demanding that a potential exists implies that the curvature scalar is a non-zero constant, the metric is Einstein and the Weyl tensor must be anti-self-dual (self-dual).

The important equations are

$$\frac{24}{R}F_{\mu\nu} = \operatorname{Vec} H_{\mu\nu} = \frac{12}{R}\operatorname{Vec} H^{\rho\sigma}R_{\rho\mu\sigma\nu}$$

$$\mathbb{1}_{2\times 2}g_{\mu\nu} = \operatorname{Re} H_{\mu\nu} = \frac{4}{R}\operatorname{Re} H^{\rho\sigma}R_{\rho\mu\sigma\nu} \tag{61}$$

$$H^{\rho\sigma}C_{\mu\rho\nu\sigma} = 0.$$

The connection for the SU(2) field is obtained by projecting out a SU(2) subgroup of the O(4) spin connection, as in Charap and Duff (1977).

$$A_\mu = \tfrac{1}{4}h_\nu(h^{\nu\dagger})_{;\mu}. \tag{62}$$

Application of these results to S^4 yields the instanton solutions of Belavin *et al* (1975), and to $\mathbb{C}P^2$ yields a self-dual field with non-integer topological charge (Gibbons and Pope 1978).

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